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LETTER TO THE EDITOR

Casimir cancellations in half an Einstein universe

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Abstract. We 'halve' an Einstein universe in two ways. For the spin-0 field we equatorially bound S^3 by S^2, while for the spin-$\frac{1}{2}$ field we factor S^3 to give the lens space S^3/Z_2. In both cases the vacuum expectation value of the stress tensor is identical to that in the complete Einstein universe.

1. Introduction

Ford (1975) was the first to consider the Casimir effect in an Einstein universe (see also Ford 1976, Dowker and Critchley 1976a, 1977, Dowker and Al'taie 1978), the particular attractiveness of this manifold arising from the exact nature of the WKB approximation on S^3 (Dowker 1971). His results for the renormalised vacuum expectation value of the stress tensor are

$$\langle \hat{T}_{00} \rangle = (480\pi^2 a^4)^{-1} \tag{1.1a}$$

$$\langle \hat{T}_{ij} \rangle = -(1440\pi^2 a^4)^{-1} g_{ij} \tag{1.1b}$$

for the conformally coupled spin-0 field, and

$$\langle \hat{T}_{00} \rangle = 17(1920\pi^2 a^4)^{-1} \tag{1.2a}$$

$$\langle \hat{T}_{ij} \rangle = -17(5760\pi^2 a^4)^{-1} g_{ij} \tag{1.2b}$$

for the neutrino field, where a is the radius of S^3 and we employ a metric $g_{\mu\nu}$ of negative signature. These equations express the physical content of the vacuum state's dependence on the topology of the manifold.

In this article we consider the effect on $\langle \hat{T}_{\mu\nu} \rangle$ of two perverse alterations of the manifold structure: we constrain the spin-0 field to satisfy Dirichlet or Neumann boundary conditions on the submanifold S^2 which bounds S^3 equatorially, while for the spin-$\frac{1}{2}$ field we factor the spatial section to give the lens space S^3/Z_2. In both instances we obtain precisely the results above for the complete Einstein universe. (Actually, since we take the massless limit of the massive spin-$\frac{1}{2}$ results, we obtain twice the neutrino values above.)

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2. Spin-0 field

This field satisfies

$$(\Box - R/6)\phi(x) = 0 \tag{2.1}$$

in $M = \mathbb{R} \otimes S^3$ subject to one of the boundary conditions

$$\text{Dirichlet:} \qquad \phi(x) = 0 \tag{2.2a}$$

$$\text{Neumann:} \qquad n^\alpha \nabla_\alpha \phi(x) = 0 \tag{2.2b}$$

on $\partial M = \mathbb{R} \otimes S^2$ where S^2 bounds S^3 equatorially and n_α is the unit inward pointing normal form on ∂M (Hawking and Ellis 1973). The corresponding Green function $G(x, x')$ satisfies

$$(\Box - R/6)G(x, x') = \delta(x, x'), \tag{2.3}$$

where $\delta(x, x')$ is the covariant delta function (Dowker and Critchley 1976b), and one of

$$\text{Dirichlet:} \qquad G(x, x') = 0 \tag{2.4a}$$

$$\text{Neumann:} \qquad n^\alpha \nabla_\alpha G(x, x') = 0 \tag{2.4b}$$

when $x \in \partial M$.

Reflecting $\{x\}$ in ∂M to give $\{\tilde{x}\}$ we obtain the double manifold $D = M \cup \partial M \cup M^*$, the complete Einstein universe, on which the Green function is expressible as the image sum (Dowker and Critchley 1976a)

$$D(x, x') = \sum_{n=-\infty}^{\infty} \frac{-\mathrm{i}}{4\pi^2 \sin\alpha} \frac{\alpha_n}{\sigma_n^2} \tag{2.5}$$

where $\alpha_n = \alpha + 2\pi n$, $a\alpha$ is the geodesic distance on S^3 between $\boldsymbol{x}$ and $\boldsymbol{x}'$ and $\sigma_n^2 = (x^0 - x^{0\prime})^2 - a^2\alpha_n^2 - \mathrm{i}\epsilon$. We can therefore satisfy (2.4) by locating an image charge at $\tilde{x}'$ in the dual M^* region (McKean and Singer 1967) giving

$$G(x, x') = D(x, x') \mp D(x, \tilde{x}') \tag{2.6}$$

where the upper (lower) sign indicates Dirichlet (Neumann) boundary conditions.

Expressing $\langle \hat{T}_{\mu\nu} \rangle$ as the coincidence limit (DeWitt 1975)

$$\begin{aligned}\langle \hat{T}_{\mu\nu} \rangle = (-\mathrm{i}/6) \lim_{x' \to x} [&4\nabla_\mu \nabla_{\nu'} + g_{\mu\nu'}(\nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'} - g^{\lambda\sigma'} \nabla_\lambda \nabla_{\sigma'}) - g_{\mu\rho'} \nabla^{\rho'} \nabla_{\nu'} \\ &- g_{\nu'\sigma} \nabla^\sigma \nabla_\mu + \tfrac{1}{2}(R_\mu{}^\sigma g_{\sigma\nu'} + R^{\rho'}{}_{\nu'} g_{\mu\rho'} - R g_{\mu\nu'})] G(x, x') \\ \equiv [&\vec{T}_{\mu\nu} G]\end{aligned} \tag{2.7}$$

we see that $\mp[\vec{T}_{\mu\nu} D(x, \tilde{x}')]$ gives the finite correction to the formally divergent expression $[\vec{T}_{\mu\nu} D(x, x')]$ for the complete Einstein universe. The calculation and regularisation of the latter quantity is performed in Dowker and Critchley (1976a) and yields equations (1.1).

To show that the correction $\mp[\vec{T}_{\mu\nu} D(x, \tilde{x}')]$ vanishes, it is convenient to separate off the $n = 0$ term in (2.5) yielding

$$[\vec{T}_{\mu\nu} D(x, \tilde{x}')] = [\vec{T}_{\mu\nu} D_{n=0}(x, \tilde{x}')] + [\vec{T}_{\mu\nu} D_{n \neq 0}(x, \tilde{x}')]. \tag{2.8}$$

The first term on the RHS gives the correction to (1.1) due to the presence of ∂M if M

were in fact *locally* $\mathbb{R}\otimes S^3$ but had no global topological features. The second term expresses the fact that multiple scattering can occur by crossing the hemi-hypersphere.

Introducing hyperspherical polars $(\theta_1, \theta_2, \phi)$ (Erdélyi *et al* 1953) such that $\theta_1 = \pi/2$ identifies ∂M, we easily obtain

$$[\vec{T}_{00}D_{n=0}(x, \tilde{x}')] = (24\pi^2 a^4)^{-1} \operatorname{cosec} 2\theta_1[(\pi - 2\theta_1)^{-1} \sec^2 \theta_1 + 2(\pi - 2\theta_1)^{-2} \tan \theta_1 - 8(\pi - 2\theta_1)^{-3}] \quad (2.9a)$$

$$[\vec{T}_{ij}D_{n=0}(x, \tilde{x}')] = (24\pi^2 a^4)^{-1} \operatorname{cosec} 2\theta_1\{[(\pi - 2\theta_1)^{-1} \operatorname{cosec}^2 \theta_1 - 2(\pi - 2\theta_1)^{-2} \cot \theta_1]n_i n_j + [2(\pi - 2\theta_1)^{-1} \cot 2\theta_1 \operatorname{cosec} 2\theta_1 - 2(\pi - 2\theta_1)^{-2} \operatorname{cosec} 2\theta_1 + 4(\pi - 2\theta_1)^{-3}]h_{ij}\} \quad (2.9b)$$

where $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the metric on ∂M (Hawking and Ellis 1973). As expected, we thus find a θ_1 dependence in $\langle\hat{T}_{\mu\nu}\rangle$. Using the summation (Gradshteyn and Ryzhik 1965)

$$\sum_{n=1}^{\infty} [n^2 - x^2]^{-1} = \frac{1}{2x^2} - \frac{\pi}{2x} \cot \pi x \quad (2.10)$$

the second term on the RHS of (2.8) may be cast in closed form. Surprisingly we find

$$[\vec{T}_{\mu\nu}D_{n\neq 0}(x, \tilde{x}')] = -[\vec{T}_{\mu\nu}D_{n=0}(x, \tilde{x}')] \quad (2.11)$$

and so

$$\langle\hat{T}_{\mu\nu}\rangle = [\vec{T}_{\mu\nu}D(x, x')]. \quad (2.12)$$

The topological aspects of the manifold, in the form of multiple scattering, exactly cancel the local θ_1 dependence, leaving only the vacuum stress (1.1) for the complete Einstein universe.

3. Spin-$\frac{1}{2}$ field in lens spaces

Spin-$\frac{1}{2}$ stress expectation values may also be expressed as the point coincidence limit of operators acting upon the Feynman Green function for the spacetime as expounded by Dowker and Al'taie (1978) in their Einstein universe calculation. This method admits a rather straightforward modification in order to perform the analagous $\mathbb{R}\otimes S^3/Z_r$ lens space calculations, r being a positive integer. Their 'renormalisation' procedure of dropping the 'direct' term in the image constructed Einstein universe massive Green function yields results which agree in the massless limit with Ford's (1976) neutrino calculations involving a field mode summation. However, when starting explicitly with the two-component theory neutrino Green function, the same renormalisation procedure leads to a distinct set of results, and this fact we use as one justification for considering the massive spin-$\frac{1}{2}$ field in lens spaces.

In order to derive an expression for the second order, 'squared' spin-$\frac{1}{2}$ Green function in $\mathbb{R}\otimes S^3/Z_r$, we note the following points.

(i) The space S^3 is invariant under left and right SU(2) transformations, that is, its symmetry group is $\mathrm{SU(2)_L}\otimes\mathrm{SU(2)_R}$.

(ii) The lens space S^3/Z_r may be expressed as $S^3/(\Gamma_L\otimes\Gamma_R)$ where $\Gamma_L = 1$, the identity element, and Γ_R is the group which generates the factor space by point identifications in S^3. We might just as well have reversed the roles of Γ_L and Γ_R but we adhere to this choice for our calculations.

(iii) Provided the local *vierbein* with respect to which the spinor field is constructed remains invariant under the action of all the $\gamma_R(\in\Gamma_R)$, the lens space second order Green function may be written down by identifying points in the covering space, that is

$$S(x, x') = \sum_{\gamma_R} S_{S^3}(x, x'\gamma_R) \tag{3.1}$$

where S_{S^3} is the Green function in $\mathbb{R}\otimes S^3$.

The Killing vectors of the left SU(2) group are invariant under all right transformations, and in particular, those belonging to Γ_R, hence they present an obvious choice as local *dreibeine* for which (3.1) is valid. The fourth *bein* points in the time direction and the resulting spinors we call 'left spinors'. It remains for us to find the explicit form of the point identifications incorporated in (3.1) and to do this, we embed S^3 in four-dimensional Euclidean space which is identified with $C\otimes C$ such that S^3 is the set of ordered pairs (ζ_1, ζ_2) satisfying

$$|\zeta_1|^2+|\zeta_2|^2=a^2. \tag{3.2}$$

S^3/Z_r is now generated by the identifications (our lens space being classified by Seifert and Threlfall (1934) as Linsenräume $(r, 1)$)

$$(\zeta_1, \zeta_2)\to(\zeta_1\exp(2\pi i/r), \zeta_2\exp(2\pi i/r)) \tag{3.3}$$

or, if parametrised in terms of the Euler angles ψ, θ and ω and choosing the pairs such that

$$\zeta_1 = a\exp[(i/2)(\psi+\omega)]\cos(\theta/2), \qquad \zeta_2 = a\exp[(i/2)(\psi-\omega)]\sin(\theta/2), \tag{3.4}$$

(3.3) may be expressed as

$$(\psi, \theta, \omega)\to(\psi+4\pi/r, \theta, \omega). \tag{3.5}$$

At this point, we reproduce Dowker and Al'taie's (1978) expression for the left spinor second order Feynman Green function in $\mathbb{R}\otimes S^3$:

$$S_{S^3}(x, x') = \sum_{n=-\infty}^{\infty} \exp[(i/2)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{x}}\alpha_n] \times\left\{\frac{\alpha_n}{\sin\alpha_n}\left[\frac{-m}{8\pi}\frac{H_1^{(2)}(m\sigma_n)}{\sigma_n}\right]-\frac{1}{8\pi a^2\cos^2(\alpha_n/2)}\left[\frac{1}{4}H_0^{(2)}(m\sigma_n)\right]\right\} \tag{3.6}$$

where σ^i are the Pauli matrices, $H_n^{(2)}$ is a Hankel function and S_{S^3} satisfies

$$-(i\gamma^\mu\nabla_\mu - m)(i\gamma^\nu\nabla_\nu + m)S_{S^3}\oplus S_{S^3}(x, x') = 1\delta(x, x') \tag{3.7}$$

with the aforementioned choice in *vierbein*. To invoke (3.1) in the construction of $S(x, x')$ we take $\boldsymbol{x}$ to be the origin ($\psi=\theta=\omega=0$) and $\boldsymbol{x}'$ to be described by $(\psi', \theta', \omega')$. $\Gamma_R=\{\gamma^n\}$ where n runs from 0 to $r-1$ and γ generates the transformation (3.5). The radial separation of $\boldsymbol{x}$ and $\boldsymbol{x}'\gamma^n$, $a\chi_n$, in terms of the Euler angles is given by

$$\cos\chi_n = \cos\frac{\theta'}{2}\cos\left(\frac{\psi'+\omega'}{2}+\frac{2\pi n}{r}\right) \tag{3.8}$$

and making the choice $\theta'=0$, we obtain

$$\chi_n = \chi_0 + 2\pi n/r. \tag{3.9}$$

$S_{S^3}(x, x')$ depends upon the radial separation of x and x', hence (3.1) now implies that the right-hand side of (3.6) with the replacement $\alpha_n \to \chi_n$ gives $S(x, x')$. In terms of this 2×2 Green function, the field energy density and stress tensor trace expectation values may be written as

$$\langle \hat{T}_{00}\rangle = -2\mathrm{i} \lim_{x'\to x} \mathrm{Tr}\, \partial_0\partial_0 S(x, x') \tag{3.10a}$$

and

$$\langle \hat{T}_\mu{}^\mu\rangle = 2\mathrm{i}m^2 \lim_{x'\to x} \mathrm{Tr}\, S(x, x'). \tag{3.10b}$$

The lens spaces being considered are homogeneous and therefore our choice in the locations of x and x' is not at the expense of positional generality in the expectation values. The remaining calculations are straightforward and we obtain expressions for the expectation values which, like S, are sums over n. Eventually we shall choose $r = 2$ but meanwhile, we merely assume r to be even and split the sums into terms for which $n = kr$, $n = (2k+1)r/2$, where k runs over all integers, and the remaining terms, these being terms for which $n \neq kr/2$. The first sum, over $n = kr$, gives the S^3 expressions and 'renormalisation' consists of dropping the $k = 0$ term, this being the only source of divergence in the lens space values. The contribution from the $n = (2k+1)r/2$ terms turns out to be zero and the remaining terms, that is, those for which $n \neq kr/2$, give the lens space correction to the Einstein universe expectation values $\langle \hat{T}_{\mu\nu}\rangle_{S^3}$ of Dowker and Al'taie (1978). We find

$$\langle \hat{T}_{00}\rangle = \langle \hat{T}_{00}\rangle_{S^3} + \frac{mr}{4\pi^3 a^3} \sum_{\substack{n=1\\ n\neq kr/2}}^{\infty} \frac{1}{n}\left[2ma\,\mathrm{cosec}\frac{n\pi}{r} K_2(2\pi manr^{-1}) - \sec\frac{n\pi}{r} K_1(2\pi manr^{-1})\right] \tag{3.11a}$$

and

$$\langle \hat{T}_\mu{}^\mu\rangle = \langle \hat{T}_\mu{}^\mu\rangle_{S^3} + \frac{m^2}{2\pi^2 a^2} \sum_{\substack{n=1\\ n\neq kr/2}}^{\infty} \left[\sec\frac{n\pi}{r} K_0(2\pi manr^{-1}) - 2ma\,\mathrm{cosec}\frac{n\pi}{r} K_1(2\pi manr^{-1})\right]. \tag{3.11b}$$

For odd r, (3.11) still holds if we replace $n \neq kr/2$ by $n \neq kr$. Putting $r = 2$ we see that no terms in the correctional summations survive and that the expectation values are identical to those in S^3. What is more, the global group of isometries of S^3/Z_2, like that of S^3, is spin (4) and hence $\langle \hat{T}_{\mu\nu}\rangle_{r=2}$ has the same tensor structure as $\langle \hat{T}_{\mu\nu}\rangle_{S^3}$, that is $\langle \hat{T}_i^0\rangle_{r=2} = 0$ and $\langle \hat{T}_i^j\rangle_{r=2} = \frac{1}{3}g_i^j(\langle \hat{T}_\mu{}^\mu\rangle_{r=2} - \langle \hat{T}_{00}\rangle_{r=2})$. We have therefore $\langle \hat{T}_{\mu\nu}\rangle_{r=2} = \langle \hat{T}_{\mu\nu}\rangle_{S^3}$.

We might also mention that an alternative method of obtaining these results is to use homogeneity to determine $\langle \hat{T}_{00}\rangle$ from the mode eigenvalues and then find $\langle \hat{T}_{ij}\rangle$ from symmetry and a knowledge of the trace $\langle \hat{T}_i^i\rangle$ (which follows from a variation of $\langle \hat{T}_{00}\rangle$ with respect to the radius, see Dowker and Critchley 1977). This method avoids point splitting and a knowledge of the Green function.

Here, we briefly remark upon the existence of twisted spinor connections on the lens spaces for which r is even. Spin connections, like real scalar fields, in these spaces are labelled by the cohomology group $H^1(S^3/Z_r, Z_2)$ and just as twisted real scalar fields exist for even r (Dowker and Banach 1978), so do twisted spin connections. The twist may be transferred from the connections onto the spinor field itself (Chockalingham and Isham 1980) and this introduces a factor of $(-1)^n$ on the left of the brackets $\{\}$ in (3.6) after the $\alpha_n \to \chi_n$ replacement, to give the twisted spinor Green function. This factor of $(-1)^n$ carries through into the summations of (3.11) and leaves the S^3/Z_2 results indistinguishable from the untwisted spinor case.

4. Discussion

In connection with the spin-0 case considered, it is perhaps worthwhile to draw attention to certain similarities to a better known situation. This is the case of a conformally invariant spin-0 field in the presence of a single plane boundary in Minkowski space where, also, the field stress expectation values are identical to those for the complete manifold, in this case zero. In both situations the scalar field is conformally invariant and S^2 bounding S^3 equatorially, in common with a plane boundary in Minkowski space, has vanishing extrinsic curvature. (Compare with Deutsch and Candelas' 1979 work on curved boundaries.)

Halving the Einstein universe to obtain the lens space S^3/Z_2 is a quite distinct procedure resulting in a multiply connected space doubly covered by S^3 and at the time of writing, the physical connection between the two situations considered in this article is not apparent to the authors. A further point to note is that although the spin-$\frac{1}{2}$ stress expectation values are insensitive to this factoring of S^3, the same is not true of the conformally invariant spin-0 field (Dowker and Banach 1978).

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